J Glob Optim (2008) 41:465–478
DOI 10.1007/s10898-007-9258-6

Mixed projection methods for systems of variational inequalities

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Received: 27 December 2005 / Accepted: 27 October 2007 / Published online: 23 November 2007

Abstract Let H be a real Hilbert space. Let $F{:}D(F) \subseteq H \to H,\ K : D(K) \subseteq H \to H$ be bounded and continuous mappings where $D(F)$ and $D(K)$ are closed convex subsets of H. We introduce and consider the following system of variational inequalities: find $[u^*, v^*] \in D(F) \times D(K)$ such that

$$\begin{cases} \langle Fu^* - v^*, x - u^* \rangle \geq 0, & x \in D(F), \\ \langle Kv^* + u^*, y - v^* \rangle \geq 0, & y \in D(K). \end{cases}$$

This system of variational inequalities is closely related to a pseudomonotone variational inequality. The well-known projection method is extended to develop a mixed projection method for solving this system of variational inequalities. No invertibility assumption is imposed on F and K. The operators K and F also need not be defined on compact subsets of H.

Keywords Hilbert space · Variational inequality · Pseudomonotonicity · Mixed projection method

Mathematics Subject Classification (2000) 49J30 · 47H10 · 47H17

1 Introductions and preliminaries

Let H be a real Hilbert space with norm and inner product denoted by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively. Let $F : D(F) \subseteq H \to H$ be a nonlinear operator, where the domain of F is a

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nonempty closed convex subset of H. The classical variational inequality problem is to find $x^* \in D(F)$ such that

$$\langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in D(F).$$

A survey on the variational inequality problem in finite-dimensional spaces was done by Harker and Pang [6]. In [6], the reader will find motivations, examples, results, and a vast bibliography. Various iterative methods have been suggested and proposed for solving variational inequalities. In particular, the projection method and its variant forms have widely been studied and applied to solving variational inequalities and various generalizations of variational inequalities; see for example [18–20].

Let A be an operator with domain and range denoted by $D(A)$ and $\mathcal{R}(A)$, respectively. In what follows, we recall some concepts which will be used in the sequel.

Definition 1.1 [3] Let $A : D(A) \subseteq H \to H$ be a mapping.

(i) A is called monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in D(A);$$

(ii) A is called maximal monotone if it is monotone and $\mathcal{R}(I + rA) = H$ for each $r > 0$, where I is the identity mapping on H and $\mathcal{R}(I + rA)$ denotes the range of $(I + rA)$;

(iii) A is said to satisfy the range condition if $\text{cl}(D(A)) \subseteq \mathcal{R}(I + rA)$ for each $r > 0$ where $\text{cl}(D(A))$ denotes the closure of the set $D(A)$;

(iv) A is called uniformly monotone if there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \phi(\|x - y\|).$$

The notion of monotone operators was introduced independently by Zarantonello [17] and Minty [9]. In Definition 1.1 (iv), if $\phi(t) = t\psi(t)$ for $\psi : [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$, ψ strictly increasing, then A is called ψ-strongly monotone; if there exists $k > 0$ such that $\phi(t) = kt^2$, then A is called strongly monotone. We have the following implications:

$$\begin{aligned}\text{strong monotonicity} &\Rightarrow \psi - \text{strong monotonicity} \ \Rightarrow \ \text{uniform monotonicity} \\ &\Rightarrow \text{monotonicity}.\end{aligned}$$

It is well known that interest in monotone mappings stems mainly from their firm connection with equations of evolution. Several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear parts possess Green's function, can be put in operator form as

$$u + KFu = 0, \tag{1.1}$$

where K and F are monotone operators (see [4] for more details).

Recently, Chidume and Zegeye [4] introduced a method that contains an auxiliary operator, defined in an appropriate real Banach space in terms of K and F which under certain conditions, is ψ-strongly monotone whenever K and F are ψ-strongly monotone and whose zeros are solutions of Eq. 1.1. Further, Chidume and Zegeye [3] employed this method to obtain an auxiliary operator that is monotone whenever K and F ψ-strongly monotone and to construct an iterative procedure that converges strongly to the solution of Eq. 1.1.

On the other hand, we first recall the concept of pseudomonotonicity. A mapping $A : D(A) \subseteq H \to H$ is called pseudomonotone if for each $x, y \in D(A)$ there holds

$$\langle Ax, y - x \rangle \geq 0 \quad \Rightarrow \quad \langle Ay, y - x \rangle \geq 0.$$

Pseudomonotonicity is understood here in the sense of Karamardian [8] and not in the sense of Brézis [1]. The latter concerns some topological properties on the operator. A mapping $A : D(A) \subseteq H \to H$ is strongly pseudomonotone [7] if there exists a constant $\kappa > 0$ such that for each $x, y \in D(A)$ there holds

$$\langle Ax, y - x\rangle \geq 0 \quad \Rightarrow \quad \langle Ay, y - x\rangle \geq \kappa\|y - x\|^2.$$

We illustrate hereafter the relationships (see [16,13]) between the monotonicity assumption and some generalized monotonicity assumptions:

$$\begin{array}{ccc} \text{strong monotonicity} & \Rightarrow & \text{monotonicity} \\ \Downarrow & & \Downarrow \\ \text{strong pseudomonotonicity} & \Rightarrow & \text{pseudomonotonicity} \end{array}$$

Let H be a real Hilbert space and let $F : D(F) \subseteq H \to H$, $K : D(K) \subseteq H \to H$ be non-linear mappings where $D(F)$ and $D(K)$ are closed convex subsets of H. Let us consider the following system of variational inequalities: find $[u^*, v^*] \in D(F) \times D(K)$ such that

$$\begin{cases} \langle Fu^* - v^*, x - u^*\rangle \geq 0, & x \in D(F), \\ \langle Kv^* + u^*, y - v^*\rangle \geq 0, & y \in D(K). \end{cases} \tag{1.2}$$

We define the set of all solutions of system (1.2) by

$$\Omega := \{[u^*, v^*] \in D(F) \times D(K) : [u^*, v^*] \text{ satisfies system (1.2)}\}$$

The following is elementary.

Proposition 1.1 [3] Let H be a real Hilbert space. Let $E := H \times H$ with norm

$$\|z\|_E := (\|u\|_H^2 + \|v\|_H^2)^{1/2}, \quad \text{where } z = [u, v].$$

Then E is a real Hilbert space and for $w_1 = [u_1, v_1]$, $w_2 = [u_2, v_2] \in E$ we have that $\langle w_1, w_2\rangle = \langle u_1, u_2\rangle + \langle v_1, v_2\rangle$.

In the sequel we shall need the following results.

Lemma 1.1 [15] Let $\{\beta_n\}_{n=0}^{\infty}$ be a sequence of non-negative real numbers with

$$\beta_{n+1} \leq (1 - \delta_n)\beta_n + \sigma_n, \quad n = 0, 1, 2, \ldots,$$

where $\delta_n \in [0, 1]$, $\sum_{n=0}^{\infty} \delta_n = \infty$ and $\sigma_n = o(\delta_n)$. Then $\lim_{n\to\infty} \beta_n = 0$.

Theorem SR [12] Let H be a real Hilbert space. Let $A \subset H \times H$ be monotone with $\text{cl}(D(A))$, the closure of $D(A)$, convex and suppose that A satisfies the range condition: $\text{cl}(D(A) \subset \mathcal{R}(I + rA), \forall r > 0$. Let $J_t x := (I + tA)^{-1}x, t > 0$ be the resolvent of A and assume that $A^{-1}(0)$ is nonempty. Then for each $x \in H$, $\lim_{t\to\infty} J_t x = P_K x \in A^{-1}(0)$ where P_K is the metric projection from $\text{cl}(D(A))$ onto $A^{-1}(0)$.

Theorem TZ [14] Let H be a Hilbert space and let $A : D(A) \subseteq H \to H$ be a maximal monotone mapping. Suppose that for some $x_0 \in D(A)$ and $r > 0$ we have $\|Ax_0\| < r \leq \liminf_{x\in D(A), \|x\|\to\infty} \|Ax\|$. Then $B_r(0) = \{x \in E : \|x\| < r\} \subseteq \mathcal{R}(A)$.

Theorem CZ [2] Suppose K is a closed convex subset of a Hilbert space H. Suppose $A : K \to H$ is a bounded uniformly monotone map and that the equation $Ax = 0$ has a solution. For arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by

$$x_{n+1} := P_K(x_n - \alpha_n Ax_n), \quad n \geq 1, \tag{1.3}$$

where $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. Then there exists a constant $d_0 > 0$ such that if $0 < \alpha_n \le d_0$, then $\{x_n\}$ converges strongly to the unique solution of $Ax = 0$.

We remark that Theorems SR, TZ and CZ are, respectively, special cases of theorems proved in more general Banach spaces in Reich [12] (Theorem 1, Remarks 1 and 2), Takahashi and Zhang [14] and Chidume and Zegeye [2] (Theorem 3.8).

Lemma 1.2 [[3] Lemma 3.1]. Let H be a real Hilbert space. Let $F : D(F) \subseteq H \to H$, $K : D(K) \subseteq H \to H$ be monotone mappings. Let $E := H \times H$ with norm $\|z\|_E^2 = \|u\|_H^2 + \|v\|_H^2$ for $z := [u, v] \in E$. Define a mapping $T : D(F) \times D(K) \to E$ by $Tz = T[u, v] := [Fu - v, u + Kv]$. Then for each $z_1, z_2 \in D(F) \times D(K)$ we have that

$$\langle Tz_1 - Tz_2, z_1 - z_2\rangle \ge 0,$$

i.e., T is monotone. Moreover, if F and K are bounded, then T is bounded.

We remark that the method of proof of Lemma 3.1 [3] also yields that if F and K are uniformly monotone, then T is uniformly monotone with $\phi := \min\{\phi_1, \phi_2\}$ where ϕ_1 and ϕ_2 are the strictly increasing functions corresponding to F and K, respectively.

Lemma 1.3 [11] Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ and $\{\delta_n\}_{n=0}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \le (1 + \delta_n)a_n + b_n, \quad n \ge 0.$$

If $\sum_{n=0}^{\infty}\delta_n < \infty$ and $\sum_{n=0}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If in addition $\{a_n\}_{n=0}^{\infty}$ has a subsequence which converges strongly to zero, then $\lim_{n\to\infty} a_n = 0$.

We shall need the following concept in the sequel.

Definition 1.2 [3] Let D_1 and D_2 be subsets of a real Hilbert space H. Let $F : D_1 \to H$ and $K : D_2 \to H$ be monotone mappings. Then D_1 and D_2 are said to satisfy property (P) if $D_1 \subseteq (I + rF)(D_1) - D_2$ and $D_2 \subseteq (I + rK)(D_2) + D_1$ for each $r > 0$.

Remark 1.1 If F and K satisfy the range condition and if $D(F)$ and $D(K)$ contain the origin, then condition (P) is satisfied. Moreover, if $D(F) = D(K) = H$ and if K, F satisfy the range condition, then condition (P) is clearly satisfied.

In 2004, Chidume and Zegeye [3] established the following important result on the iterative approximation of solutions to equation (1.1).

Theorem 1.1 [3] Let H be a real Hilbert space. Let $F : D(F) \subseteq H \to H$, $K : D(K) \subseteq H \to H$ be bounded monotone mappings with $\mathcal{R}(F) \subseteq D(K)$ where $D(F)$ and $D(K)$ are closed convex subsets of H satisfying property (P). Suppose that the equation $0 = u + KFu$ has a solution in $D(F)$. Let $\{\lambda_n\}$ and $\{\theta_n\}$ be real sequences in (0, 1] satisfying the following conditions:

(i) $\lim_{n\to\infty}\theta_n = 0$;
(ii) $\sum_{n=1}^{\infty}\lambda_n\theta_n = \infty$, $\lim_{n\to\infty}\lambda_n/\theta_n = 0$;
(iii) $\lim_{n\to\infty}(\frac{\theta_{n-1}}{\theta_n} - 1)/(\lambda_n\theta_n) = 0$. Let sequences $\{u_n\} \subseteq D(F)$ and $\{v_n\} \subseteq D(K)$ be generated from $u_0 \in D(F)$ and $v_0 \in D(K)$, respectively, by

$$u_{n+1} = P_{D(F)}(u_n - \lambda_n(Fu_n - v_n + \theta_n(u_n - w_1))),$$
$$v_{n+1} = P_{D(K)}(v_n - \lambda_n(Kv_n + u_n + \theta_n(v_n - w_2))),$$

where $w_1 \in D(F)$, $w_2 \in D(K)$ are arbitrary but fixed. Then there exists $d > 0$ such that if $\lambda_n \le d$ and $\lambda_n/\theta_n \le d^2$ for all $n \ge 0$, the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to u^* and v^*, respectively, in H, where u^* is a solution of the equation $0 = u + KFu$ and $v^* = Fu^*$.

2 Main results

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, respectively. Following Lemma 1.2, we let $E := H \times H$ be endowed with norm $\|z\|_E^2 = \|u\|_H^2 + \|v\|_H^2$ for $z = [u, v] \in E$. Moreover, we define an auxiliary operator $T : D(T) = D(F) \times D(K) \to E$ by

$$Tz = T[u, v] := [Fu - v, u + Kv], \quad \forall z = [u, v] \in D(F) \times D(K).$$

Theorem 2.1 Let H be a real Hilbert space. Let $F : D(F) \subseteq H \to H$, $K : D(K) \subseteq H \to H$ be bounded continuous mappings such that $T : D(T) \to E$ is pseudomonotone where $D(F)$ and $D(K)$ are closed convex subsets of H. Suppose that $\Omega \neq \emptyset$. Let $\{\alpha_n\}$, $\{\lambda_n\}$ and $\{\theta_n\}$ be real sequences in $(0, 1]$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \theta_n = 0$;
(ii) $\sum_{n=1}^{\infty} \lambda_n\theta_n < \infty$, $\lim_{n\to\infty} \lambda_n/\theta_n = 0$;

Let sequences $\{u_n\} \subseteq D(F)$ and $\{v_n\} \subseteq D(K)$ be generated from $u_0 \in D(F)$ and $v_0 \in D(K)$, respectively, by

$$\begin{cases} u_{n+1} = (1 - \alpha_n)u_n + \alpha_n P_{D(F)}(u_n - \lambda_n(Fu_n - v_n + \theta_n(u_n - w_1))), \\ v_{n+1} = (1 - \alpha_n)v_n + \alpha_n P_{D(K)}(v_n - \lambda_n(Kv_n + u_n + \theta_n(v_n - w_2))), \end{cases} \tag{2.1}$$

where $w_1 \in D(F)$, $w_2 \in D(K)$ are arbitrary but fixed. Then there exists $d > 0$ such that whenever $\lambda_n \le d$ and $\lambda_n/\theta_n \le d^2$ for all $n \ge 0$, $\{z_n\}$ converges strongly to an element of Ω if and only if $\liminf_{n\to\infty} d(z_n, \Omega) = 0$, where $z_n = [u_n, v_n]$, $\forall n \ge 0$ and $d(x, C)$ denotes the distance of x to the set C in H.

Proof Recall that there holds the following equality

$$\|\lambda z_1 + (1 - \lambda)z_2\|^2 = \lambda\|z_1\|^2 + (1 - \lambda)\|z_2\|^2 - \lambda(1 - \lambda)\|z_1 - z_2\|^2$$

for all $z_1, z_2 \in H$ and $0 \le \lambda \le 1$.

"Necessity." Suppose $\{z_n\}$ converges strongly to an element $z^* = [u^*, v^*]$ of Ω. Then we derive

$$d(z_n, \Omega) \le d(z_n, z^*) = \|z_n - z^*\| \to 0, \quad \text{as } n \to \infty.$$

Hence,

$$\liminf_{n\to\infty} d(z_n, \Omega) = 0.$$

"Sufficiency." Suppose $\liminf_{n\to\infty} d(z_n, \Omega) = 0$. We divide the remainder of the proof into several steps below.

Step 1. We claim that the sequences $\{u_n\}$ and $\{v_n\}$ conforming to (2.1) are well defined. Indeed for initial point $z_0 := [u_0, v_0]$, define the sequence $\{z_n\}$ by

$$z_{n+1} := (1 - \alpha_n)z_n + \alpha_n P_{D(F)\times D(K)}(z_n - \lambda_n(Tz_n + \theta_n(z_n - w))), \tag{2.2}$$

for arbitrary but fixed $w = [w_1, w_2]$. One can show that $z_{n+1} = [u_{n+1}, v_{n+1}]$. In fact, it suffices to show that $P_{D(F)\times D(K)}[u, v] = [P_{D(F)}u, P_{D(K)}v]$. Observe that

$$\begin{aligned}
\|P_{D(F)\times D(K)}[u, v] - [u, v]\|_E^2 &= \min_{[x,y]\in D(F)\times D(K)} \|[u, v] - [x, y]\|_E^2 \\
&= \min_{[x,y]\in D(F)\times D(K)} \{\|u - x\|_H^2 + \|v - y\|_H^2\} \\
&= \|P_{D(F)}u - u\|_H^2 + \|P_{D(K)}v - v\|_H^2 \\
&= \|[P_{D(F)}u - u, P_{D(K)}v - v]\|_E^2 \\
&= \|[P_{D(F)}u, P_{D(K)}v] - [u, v]\|_E^2
\end{aligned} \tag{2.3}$$

(equality (2.3) follows since $\|\cdot\|^2$ is a continuous and convex function). This implies that $\|P_{D(F)\times D(K)}[u, v] - [u, v]\| = \|[P_{D(F)}u, P_{D(K)}v] - [u, v]\|$. Then by the uniqueness of the projection we deduce that $P_{D(F)\times D(K)}[u, v] = [P_{D(F)}u, P_{D(K)}v]$. Consequently, (2.1) is equivalent to (2.2). Hence this shows that the sequences $\{u_n\}$ and $\{v_n\}$ are well defined.

Step 2. We claim that the sequence $\{z_n\}$ is bounded. Indeed, let $z^* = [u^*, v^*]$ be a solution of system (1.2). Then we have that

$$\begin{cases} \langle Fu^* - v^*, u - u^*\rangle \geq 0, & u \in D(F), \\ \langle Kv^* + u^*, v - v^*\rangle \geq 0, & v \in D(K), \end{cases}$$

which implies that

$$\langle Tz^*, z - z^*\rangle \geq 0, \quad \forall z = [u, v] \in D(T).$$

Since $T : D(T) \to E$ is pseudomonotone, we obtain that

$$\langle Tz, z - z^*\rangle \geq 0, \quad \forall z = [u, v] \in D(T).$$

Let $r > 1$ be sufficiently large such that $z_0 \in B_r(z^*)$ and $w \in B_{\frac{r}{2}}(z^*)$. Set

$$M := [2r + \sup\{\|Tz_n\| : z_n \in \overline{B_r(z^*)}\}]^2.$$

In order to prove the boundedness of $\{z_n\}$, it suffices to show that $\{z_n\}$ is a sequence in $B = \overline{B_r(z^*)}$. We do this by induction. Note that $z_0 \in B$ by assumption. Hence we may assume $z_n \in B$. In order to prove that $z_{n+1} \in B$, suppose to the contrary z_{n+1} is not in B. Then $\|z_{n+1} - z^*\| > r$ and thus from (2.2) we have that

$$\begin{aligned}
\|z_{n+1} - z^*\| &\leq (1 - \alpha_n)\|z_n - z^*\| \\
&\quad + \alpha_n\|P_{D(F)\times D(K)}(z_n - \lambda_n(Tz_n + \theta_n(z_n - w))) - z^*\| \\
&\leq (1 - \alpha_n)\|z_n - z^*\| \\
&\quad + \alpha_n\|z_n - \lambda_n(Tz_n + \theta_n(z_n - w)) - z^*\| \\
&\leq \|z_n - z^*\| + \alpha_n\lambda_n\|Tz_n\| + \alpha_n\lambda_n\theta_n\|z_n - w\| \\
&\leq r + \sqrt{M}.
\end{aligned}$$

Moreover, from (2.2) and the fact that $\theta_n \leq 1$, we get that

$$\begin{aligned}
\|z_{n+1} - z^*\|^2 &\leq (1 - \alpha_n)\|z_n - z^*\|^2 \\
&\quad + \alpha_n\|z_n - z^* - \lambda_n(Tz_n + \theta_n(z_n - w))\|^2 \\
&\leq (1 - \alpha_n)\|z_n - z^*\|^2 + \alpha_n[\|z_n - z^*\|^2 \\
&\quad - 2\lambda_n\langle Tz_n, z_n - z^*\rangle - 2\lambda_n\theta_n\langle z_n - w, z_n - z^*\rangle \\
&\quad + \lambda_n^2\|Tz_n + \theta_n(z_n - w)\|^2] \\
&\leq \|z_n - z^*\|^2 - 2\alpha_n\lambda_n\langle Tz_n, z_n - z^*\rangle \\
&\quad - 2\alpha_n\lambda_n\theta_n\langle z_n - w, z_n - z^*\rangle \\
&\quad + \alpha_n\lambda_n^2[\|Tz_n\| + \theta_n\|z_n - w\|]^2 \\
&\leq \|z_n - z^*\|^2 - 2\alpha_n\lambda_n\theta_n\langle z_n - w, z_n - z^*\rangle + \alpha_n\lambda_n^2 M.
\end{aligned} \tag{2.4}$$

Let $\kappa > 0$ be sufficiently small such that $\kappa \leq \frac{r^2}{4}\left(\frac{\sqrt{2}}{(1+\sqrt{2})\sqrt{M}}\right)^2$ and let $d := \sqrt{\kappa}$. Then since $\|z_{n+1} - z^*\| > \|z_n - z^*\|$ by our assumption, from (2.4) we conclude that

$$2\alpha_n\lambda_n\theta_n\langle z_n - w, z_n - z^*\rangle \leq \alpha_n\lambda_n^2 M,$$

and hence

$$\langle z_n - w, z_n - z^*\rangle \leq \frac{\lambda_n M}{2\theta_n} \leq \frac{M}{2}\kappa,$$

(since $\frac{\lambda_n}{\theta_n} \leq \kappa = d^2, \forall n \geq 0$ by our assumption). Now adding $\langle w - z^*, z_n - z^*\rangle$ to both sides of this inequality we get that

$$\begin{aligned}\|z_n - z^*\|^2 &\leq \tfrac{M}{2}\kappa + \langle w - z^*, z_n - z^*\rangle\\ &\leq \tfrac{M}{2}\kappa + \|w - z^*\|\|z_n - z^*\|\\ &\leq \tfrac{M}{2}\kappa + \tfrac{r}{2}\|z_n - z^*\|.\end{aligned}$$

Solving this quadratic inequality for $\|z_n - z^*\|$ and using the estimate

$$\sqrt{\frac{r^2}{16} + \frac{M}{2}\kappa} \leq \frac{r}{4} + \sqrt{\frac{M}{2}\kappa},$$

we obtain that

$$\|z_n - z^*\| \leq \frac{r}{2} + \sqrt{\frac{M}{2}\kappa}.$$

But in any case,

$$\begin{aligned}\|z_{n+1} - z^*\| &\leq (1-\alpha_n)\|z_n - z^*\|\\ &\quad + \alpha_n\|z_n - z^* - \lambda_n(Tz_n + \theta_n(z_n - w))\|\\ &\leq \|z_n - z^*\| + \alpha_n\lambda_n\|Tz_n + \theta_n(z_n - w)\|\\ &\leq \|z_n - z^*\| + \lambda_n(\|Tz_n\| + \theta_n\|z_n - w\|)\\ &\leq \tfrac{r}{2} + \sqrt{\tfrac{M}{2}\kappa} + \lambda_n\sqrt{M}\\ &\leq \tfrac{r}{2} + \tfrac{1+\sqrt{2}}{\sqrt{2}}\sqrt{\kappa M}\\ &\leq \tfrac{r}{2} + \tfrac{1+\sqrt{2}}{\sqrt{2}} \cdot \tfrac{r}{2}\tfrac{\sqrt{2}}{(1+\sqrt{2})\sqrt{M}} \cdot \sqrt{M}\\ &= \tfrac{r}{2} + \tfrac{r}{2} = r,\end{aligned}$$

by the original choices of κ and λ_n, and this contradicts the assumption that z_{n+1} is not in B. Consequently, $z_{n+1} \in B$ and hence $\{z_n\}$ lies in B. This shows that $\{z_n\}$ is bounded.

Step 3. We claim that $\lim_{n\to\infty} d(z_n, \Omega) = 0$. Indeed, let $\hat{z} = [\hat{u}, \hat{v}] \in \Omega$ be arbitrary but fixed. Utilizing the same method as in the reasoning of (2.4), we deduce that

$$\begin{aligned}\|z_{n+1} - \hat{z}\|^2 &\le \|z_n - \hat{z}\|^2 - 2\lambda_n\alpha_n\langle Tz_n, z_n - \hat{z}\rangle \\ &\quad - 2\alpha_n\lambda_n\theta_n\langle z_n - w, z_n - \hat{z}\rangle \\ &\quad + \alpha_n\lambda_n^2[\|Tz_n\| + \theta_n\|z_n - w\|]^2 \\ &\le \|z_n - \hat{z}\|^2 - 2\alpha_n\lambda_n\theta_n\langle z_n - w, z_n - \hat{z}\rangle \\ &\quad + \alpha_n\lambda_n^2[\|Tz_n\| + \theta_n\|z_n - w\|]^2 \\ &\le \|z_n - \hat{z}\|^2 + \lambda_n\theta_n \cdot 2\|z_n - w\|\|z_n - \hat{z}\| \\ &\quad + \lambda_n^2[\|Tz_n\| + \|z_n - w\|]^2 \\ &\le \|z_n - \hat{z}\|^2 + \lambda_n\theta_n[\|z_n - w\|^2 + \|z_n - \hat{z}\|^2] \\ &\quad + \lambda_n^2[\|Tz_n\| + \|z_n - w\|]^2 \\ &\le (1 + \lambda_n\theta_n)\|z_n - \hat{z}\|^2 + \lambda_n\theta_n\|z_n - w\|^2 \\ &\quad + \lambda_n^2[\|Tz_n\| + \|z_n - w\|]^2 \\ &\le (1 + \lambda_n\theta_n)\|z_n - \hat{z}\|^2 + (\lambda_n^2 + \lambda_n\theta_n)[\|Tz_n\| + \|z_n - w\|]^2 \\ &= (1 + \lambda_n\theta_n)\|z_n - \hat{z}\|^2 + (\lambda_n^2 + \lambda_n\theta_n)M_0,\end{aligned} \tag{2.5}$$

where $M_0 := \sup\{[\|Tz_n\| + \|z_n - w\|]^2 + 1 : n \ge 0\} < \infty$ (since T is bounded, and $\{z_n\}$ is bounded by Step 2). Also since $\lim_{n\to\infty} \lambda_n/\theta_n = 0$, it is clear that there exists an integer $n_0 \ge 0$ such that $\lambda_n < \theta_n, \forall n \ge n_0$. Thus, according to the condition $\sum_{n=0}^{\infty} \lambda_n\theta_n < \infty$, we conclude that $\sum_{n=0}^{\infty} \lambda_n^2 < \infty$, and hence $\sum_{n=0}^{\infty}(\lambda_n^2 + \lambda_n\theta_n)M_0 < \infty$. Now, taking the infimum over all $\hat{z} = [\hat{u}, \hat{v}] \in \Omega$, we deduce from (2.5) that

$$[d(z_{n+1}, \Omega)]^2 \le (1 + \lambda_n\theta_n)[d(z_n, \Omega)]^2 + (\lambda_n^2 + \lambda_n\theta_n)M_0. \tag{2.6}$$

Consequently, $\lim_{n\to\infty} d(z_n, \Omega)$ exists by Lemma 1.3. This shows that

$$\lim_{n\to\infty} d(z_n, \Omega) = \liminf_{n\to\infty} d(z_n, \Omega) = 0.$$

Step 4. We claim that $\{z_n\}$ is a Cauchy sequence in $D(T)$. Indeed, put $\delta_n := (\lambda_n^2 + \lambda_n\theta_n)M_0$ for all $n \ge 0$. From (2.5) we derive for each $n \ge 0$

$$\|z_{n+1} - \hat{z}\|^2 \le (1 + \delta_n)\|z_n - \hat{z}\|^2 + \delta_n, \quad \forall \hat{z} \in \Omega, \tag{2.7}$$

where $\sum_{n=0}^{\infty} \delta_n < \infty$. Now put

$$\widetilde{M} = \prod_{n=0}^{\infty}(1 + \delta_n),$$

then $1 \le \widetilde{M} < \infty$. Since $\lim_{n\to\infty} d(z_n, \Omega) = 0$ by Step 3, for arbitrary $\varepsilon > 0$ there exists an integer $n_1 \ge 0$ such that

$$d(z_n, \Omega) < \varepsilon/\sqrt{8\widetilde{M}}, \quad \forall n \ge n_1.$$

Furthermore, $\sum_{n=0}^{\infty} \delta_n < \infty$ implies that there exists an integer $n_2 \ge 0$ such that $\sum_{j=n_2}^{\infty} \delta_j < \varepsilon^2/(8\widetilde{M}), \forall n \ge n_2$. Choose $N_0 = \max\{n_1, n_2\}$.

Observe that (2.7) yields

$$\begin{aligned}\|z_{n+1}-\hat{z}\|^2 &\le (1+\delta_n)(1+\delta_{n-1})\|z_{n-1}-\hat{z}\|^2+(1+\delta_n)\delta_{n-1}+\delta_n\\ &\le \prod_{j=N_0}^{n}(1+\delta_j)\|z_{N_0}-\hat{z}\|^2+\sum_{j=N_0}^{n-1}\delta_j\prod_{i=j+1}^{n}(1+\delta_i)+\delta_n\\ &\le \widetilde{M}[\|z_{N_0}-\hat{z}\|^2+\sum_{j=N_0}^{n}\delta_j].\end{aligned} \tag{2.8}$$

Note that $d(z_{N_0},\Omega)<\frac{\varepsilon}{\sqrt{8\widetilde{M}}}$ and $\sum_{j=N_0}^{\infty}\delta_j<\frac{\varepsilon^2}{8\widetilde{M}}$. Thus for all $n,m\ge N_0$ and all $\hat{z}\in\Omega$ we have from (2.8)

$$\begin{aligned}\|z_n-z_m\|^2 &\le [\|z_n-\hat{z}\|+\|z_m-\hat{z}\|]^2\\ &\le 2\|z_n-\hat{z}\|^2+2\|z_m-\hat{z}\|^2\\ &\le 2\widetilde{M}[\|z_{N_0}-\hat{z}\|^2+\sum_{j=N_0}^{n}\delta_j]+2\widetilde{M}[\|z_{N_0}-\hat{z}\|^2+\sum_{j=N_0}^{m}\delta_j]\\ &\le 4\widetilde{M}[\|z_{N_0}-\hat{z}\|^2+\sum_{j=N_0}^{\infty}\delta_j]\\ &\le 4\widetilde{M}[\|z_{N_0}-\hat{z}\|^2+\tfrac{\varepsilon^2}{8\widetilde{M}}].\end{aligned} \tag{2.9}$$

Taking the infimum over all $\hat{z}\in\Omega$, we obtain

$$\|z_n-z_m\|^2\le 4\widetilde{M}\left([d(z_{N_0},\Omega)]^2+\frac{\varepsilon^2}{8\widetilde{M}}\right)\le 4\widetilde{M}\left(\frac{\varepsilon^2}{8\widetilde{M}}+\frac{\varepsilon^2}{8\widetilde{M}}\right)=\varepsilon^2,$$

and hence $\|z_n-z_m\|\le\varepsilon$. This shows that $\{z_n\}_{n=0}^{\infty}$ is a Cauchy sequence in $D(T)$.

Step 5. We claim that $\{z_n\}$ converges strongly to an element of Ω. Indeed, note that $\{z_n\}_{n=0}^{\infty}$ is a Cauchy sequence in $D(T)$ by Step 4. Let $\lim_{n\to\infty}z_n=\bar{z}\in D(T)$ (since $D(T)$ is closed), where $\bar{z}=[\bar{u},\bar{v}]$. Since $F: D(F)\subseteq H\to H$ and $K: D(K)\subseteq H\to H$ are continuous mappings, it is easy to verify that Ω is closed. Therefore, from the fact that $\lim_{n\to\infty}d(z_n,\Omega)=0$, we must have that $\bar{z}\in\Omega$.

Theorem 2.2 Let H be a real Hilbert space. Let $F: D(F)\subseteq H\to H$, $K: D(K)\subseteq H\to H$ be bounded mappings such that $T: D(T)\to E$ is strongly pseudomonotone with constant $\kappa>0$ where $D(F)$ and $D(K)$ are closed convex subsets of H. Suppose that $\Omega\ne\emptyset$. Let $\{\alpha_n\}$, $\{\lambda_n\}$ and $\{\theta_n\}$ be real sequences in (0, 1] satisfying the following conditions:

(i) $\lim_{n\to\infty}\theta_n=0$;
(ii) $\sum_{n=0}^{\infty}\alpha_n\lambda_n=\infty$, $\lim_{n\to\infty}\lambda_n/\theta_n=0$;

Let sequences $\{u_n\}\subseteq D(F)$ and $\{v_n\}\subseteq D(K)$ be generated from $u_0\in D(F)$ and $v_0\in D(K)$, respectively, by

$$\begin{cases} u_{n+1}=(1-\alpha_n)u_n+\alpha_n P_{D(F)}(u_n-\lambda_n(Fu_n-v_n+\theta_n(u_n-w_1))),\\ v_{n+1}=(1-\alpha_n)v_n+\alpha_n P_{D(K)}(v_n-\lambda_n(Kv_n+u_n+\theta_n(v_n-w_2))),\end{cases} \tag{2.1}$$

where $w_1\in D(F)$, $w_2\in D(K)$ are arbitrary but fixed. Then there exists $d>0$ such that whenever $\lambda_n\le d$ and $\lambda_n/\theta_n\le d^2$ for all $n\ge 0$, $\{z_n\}$ converges strongly to the unique solution of system (1.2), where $z_n=[u_n,v_n]$ for all $n\ge 0$.

Proof Let $\bar{z}$ and $\hat{z}$ be two arbitrary elements in Ω with $\bar{z} = [\bar{u}, \bar{v}]$ and $\hat{z} = [\hat{u}, \hat{v}]$, where $\bar{u}, \hat{u} \in D(F)$ and $\bar{v}, \hat{v} \in D(K)$. Then $\bar{z}, \hat{z} \in D(T)$. Moreover, we have

$$\begin{cases} \langle F\bar{u} - \bar{v}, \hat{u} - \bar{u} \rangle \geq 0, \\ \langle K\bar{v} + \bar{u}, \hat{v} - \bar{v} \rangle \geq 0, \end{cases}$$

$$\begin{cases} \langle F\hat{u} - \hat{v}, \bar{u} - \hat{u} \rangle \geq 0, \\ \langle K\hat{v} + \hat{u}, \bar{v} - \hat{v} \rangle \geq 0. \end{cases}$$

Consequently, we obtain

$$\langle T\bar{z}, \hat{z} - \bar{z} \rangle \geq 0, \tag{2.10}$$

$$\langle T\hat{z}, \bar{z} - \hat{z} \rangle \geq 0. \tag{2.11}$$

Since T is strongly pseudomonotone with constant $\kappa > 0$, hence from (2.10) and (2.11) it follows that

$$0 \geq \langle T\hat{z}, \hat{z} - \bar{z} \rangle \rangle \geq \kappa \|\hat{z} - \bar{z}\|^2.$$

Thus $\hat{z} = \bar{z}$. This implies that Ω is a singleton. Let $\Omega = \{\hat{z}\}$.

Next, we divide the remainder of the proof into several steps.

Step 1. As in Step 1 of the proof of Theorem 2.1, we can prove that the sequences $\{u_n\}$ and $\{v_n\}$ conforming to (2.1) are well defined.

Step 2. As in Step 2 of the proof of Theorem 2.1, we can prove that the sequence $\{z_n\}$ is bounded.

Step 3. We claim that the sequence $\{z_n\}$ converges strongly to the unique solution $\hat{z}$ of system (1.2). Indeed, utilizing the same method as in the reasoning of (2.4), we deduce that

$$\begin{aligned} \|z_{n+1} - \hat{z}\|^2 &\leq \|z_n - \hat{z}\|^2 - 2\alpha_n\lambda_n \langle Tz_n, z_n - \hat{z} \rangle \\ &\quad - 2\alpha_n\lambda_n\theta_n \langle z_n - w, z_n - \hat{z} \rangle \\ &\quad + \alpha_n\lambda_n^2 [\|Tz_n\| + \theta_n \|z_n - w\|]^2 \\ &\leq (1 - 2\kappa\alpha_n\lambda_n)\|z_n - \hat{z}\|^2 - 2\alpha_n\lambda_n\theta_n \langle z_n - w, z_n - \hat{z} \rangle \\ &\quad + \alpha_n\lambda_n^2 [\|Tz_n\| + \|z_n - w\|]^2 \\ &\leq (1 - 2\kappa\alpha_n\lambda_n)\|z_n - \hat{z}\|^2 + 2\alpha_n\lambda_n\theta_n \|z_n - w\| \|z_n - \hat{z}\| \\ &\quad + \alpha_n\lambda_n^2 [\|Tz_n\| + \|z_n - w\|]^2 \\ &\leq (1 - 2\kappa\alpha_n\lambda_n)\|z_n - \hat{z}\|^2 + \alpha_n\lambda_n\theta_n \hat{M} + \alpha_n\lambda_n^2 \hat{M}, \end{aligned} \tag{2.12}$$

for some constant $\hat{M} > 0$ (since $\{z_n\}$ and $\{Tz_n\}$ are bounded). Since $\sum_{n=0}^{\infty} 2\kappa\alpha_n\lambda_n = \infty$, and

$$\lim_{n\to\infty} \frac{\alpha_n\lambda_n\theta_n\hat{M} + \alpha_n\lambda_n^2\hat{M}}{2\kappa\alpha_n\lambda_n} = \lim_{n\to\infty} \frac{\hat{M}}{2\kappa}(\theta_n + \lambda_n) = 0.$$

Thus, by Lemma 1.1 we know that $\{z_n\}$ converges strongly to the unique solution $\hat{z}$ of system (1.2). □

Theorem 2.3 Let H be a real Hilbert space. Let $F : D(F) \subseteq H \to H$, $K : D(K) \subseteq H \to H$ be bounded monotone mappings with $\mathcal{R}(F) \subseteq D(K)$, where $D(F)$ and $D(K)$ are closed convex subsets of H satisfying property (P). Suppose that $N(T) := \{z^* \in D(T) : Tz^* = 0\} \neq \emptyset$. Let $\{\alpha_n\}$, $\{\lambda_n\}$ and $\{\theta_n\}$ be real sequences in $(0, 1]$ satisfying the following conditions:

(i) $\lim_{n\to\infty}\theta_n = 0$;
(ii) $\sum_{n=0}^{\infty}\alpha_n\lambda_n\theta_n = \infty$, $\lim_{n\to\infty}\lambda_n/\theta_n = 0$;
(iii) $\lim_{n\to\infty}\frac{(\frac{\theta_{n-1}}{\theta_n}-1)}{\alpha_n\lambda_n\theta_n} = 0$.

Let sequences $\{u_n\} \subseteq D(F)$ and $\{v_n\} \subseteq D(K)$ be generated from $u_0 \in D(F)$ and $v_0 \in D(K)$, respectively, by

$$\begin{cases} u_{n+1} = (1-\alpha_n)u_n + \alpha_n P_{D(F)}(u_n - \lambda_n(Fu_n - v_n + \theta_n(u_n - w_1))), \\ v_{n+1} = (1-\alpha_n)v_n + \alpha_n P_{D(K)}(v_n - \lambda_n(Kv_n + u_n + \theta_n(v_n - w_2))), \end{cases} \tag{2.1}$$

where $w_1 \in D(F)$, $w_2 \in D(K)$ are arbitrary but fixed. Then there exists $d > 0$ such that whenever $\lambda_n \le d$ and $\lambda_n/\theta_n \le d^2$ for all $n \ge 0$, $\{z_n\}$ converges strongly to an element $z^* = [u^*, v^*]$ of Ω with $Tz^* = 0$, where $z_n = [u_n, v_n]$ for all $n \ge 0$.

Proof Since K and F are bounded monotone mappings, we have by Lemma 1.2 that T is a bounded monotone mapping. Moreover, since $D(F)$ and $D(K)$ satisfy property (P), we have that T satisfies the range condition. Furthermore, we observe that if $z^* = [u^*, v^*] \in N(T)$ then $z^* \in \Omega$, i.e., $\Omega \ne \emptyset$ and that the monotonicity of T implies the pseudomonotonicity of T.

Next, we divide the remainder of the proof into several steps.

Step 1. As in Step 1 of the proof of Theorem 2.1, we can prove that the sequences $\{u_n\}$ and $\{v_n\}$ conforming to (2.1) are well defined.

Step 2. As in Step 2 of the proof of Theorem 2.1, we can prove that the sequence $\{z_n\}$ is bounded.

Step 3. We claim that $\{z_n\}$ converges strongly to an element $z^* = [u^*, v^*]$ of Ω with $Tz^* = 0$. Indeed, since T satisfies the range condition, so does $\theta^{-1}T$ for $\theta > 0$. Thus for each $n \ge 0$ there exists a unique $y_n \in D(T)$ such that

$$y_n = (I + \frac{1}{\theta_n}T)^{-1}(w). \tag{2.13}$$

This together with (2.2), yields that

$$\begin{aligned} &\|z_{n+1} - y_n\|^2 \\ &\le (1-\alpha_n)\|z_n - y_n\|^2 + \alpha_n\|z_n - y_n - \lambda_n(Tz_n + \theta_n(z_n - w))\|^2 \\ &= (1-\alpha_n)\|z_n - y_n\|^2 + \alpha_n[\|z_n - y_n\|^2 - 2\lambda_n\langle Tz_n + \theta_n(z_n - w), z_n - y_n\rangle \\ &\quad + \lambda_n^2\|Tz_n + \theta_n(z_n - w)\|^2] \\ &\le (1 - 2\alpha_n\lambda_n\theta_n)\|z_n - y_n\|^2 - 2\alpha_n\lambda_n\langle Tz_n + \theta_n(y_n - w), z_n - y_n\rangle \\ &\quad + \alpha_n\lambda_n^2\|Tz_n + \theta_n(z_n - w)\|^2. \end{aligned} \tag{2.14}$$

Moreover, since from (2.13) we have that $\theta_n(w - y_n) = Ty_n$, we obtain that $\langle Tz_n + \theta_n(y_n - w), z_n - y_n\rangle \ge 0$. Furthermore, $\{z_n\}$ and hence $\{Tz_n\}$ are bounded. The sequence $\{y_n\}$ is also bounded, because it is convergent by Theorem SR. Thus, there exists $M_1 > 0$ such that $\|Tz_n + \theta_n(z_n - w)\|^2 \le M_1$. Therefore, inequality (2.14) gives

$$\|z_{n+1} - y_n\|^2 \le (1 - 2\alpha_n\lambda_n\theta_n)\|z_n - y_n\|^2 + M_1\alpha_n\lambda_n^2. \tag{2.15}$$

On the other hand, by the monotonicity of T we have that

$$\begin{aligned} \|y_{n-1} - y_n\| &\le \|y_{n-1} - y_n + \tfrac{1}{\theta_n}(Ty_{n-1} - Ty_n)\| \\ &\le \tfrac{\theta_{n-1}-\theta_n}{\theta_n}(\|y_{n-1}\| + \|w\|) = (\tfrac{\theta_{n-1}}{\theta_n} - 1)(\|y_{n-1}\| + \|w\|). \end{aligned}$$

Therefore, this estimate together with (2.15) yields that

$$\|z_{n+1} - y_n\|^2 \le (1 - 2\alpha_n\lambda_n\theta_n)\|z_n - y_{n-1}\|^2 + M_2\left(\frac{\theta_{n-1}}{\theta_n} - 1\right) + M_2\alpha_n\lambda_n^2,$$

for some constant $M_2 > 0$. Thus by Lemma 1.1, $z_{n+1} - y_n \to 0$. Hence, since by Theorem SR, $y_n \to z^* = [u^*, v^*] \in N(T)$, we have $z_n \to z^*$. □

Theorem 2.4 Let H be a real Hilbert space. Let $F, K : H \to H$ be bounded monotone mappings. Suppose there exist $u^0, v^0 \in H$ such that

$$\|F(u^0) - v^0\|^2 + \|Kv^0 + u^0\|^2 < r \le \liminf_{u,v\in H, \|u\|,\|v\|\to\infty} (\|Fu - v\|^2 + \|Kv + u\|^2),$$

for some $r > 0$. Let $\{\alpha_n\}$, $\{\lambda_n\}$ and $\{\theta_n\}$ be real sequences in (0, 1] satisfying the following conditions:

(i) $\lim_{n\to\infty} \theta_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n\lambda_n\theta_n = \infty$, $\lim_{n\to\infty} \lambda_n/\theta_n = 0$;
(iii) $\lim_{n\to\infty} \frac{(\frac{\theta_{n-1}}{\theta_n} - 1)}{\alpha_n\lambda_n\theta_n} = 0$.

Let $u_0, v_0 \in H$ be arbitrary, and let sequences $\{u_n\}$ and $\{v_n\}$ be generated from u_0 and v_0, respectively, by

$$\begin{cases} u_{n+1} = u_n - \alpha_n\lambda_n(Fu_n - v_n + \theta_n(u_n - w_1)), \\ v_{n+1} = v_n - \alpha_n\lambda_n(Kv_n + u_n + \theta_n(v_n - w_2)), \end{cases} \tag{2.16}$$

where $w \in H$. Then there exists $d > 0$ such that whenever $\lambda_n \le d$ and $\lambda_n/\theta_n \le d^2$ for all $n \ge 0$, $\{z_n\}$ converges strongly to an element $z^* = [u^*, v^*]$ of Ω with $Tz^* = 0$, where $z_n = [u_n, v_n]$ for all $n \ge 0$.

Proof Since F and K are bounded continuous monotone mappings, it is known that $T : E \to E$ defined by $Tz = T[u, v] := [Fu - v, u + Kv]$ is also a bounded continuous monotone mapping and hence by Theorem 2 of [10], T satisfies the range condition. Clearly, T is maximal monotone. Moreover, for $z^0 = [u^0, v^0] \in E$,

$$\begin{aligned} \|T(z^0)\| = \|T[u^0, v^0]\| &= (\|F(u^0) - v^0\|^2 + \|Kv^0 + u^0\|^2)^{1/2} < r^{1/2} \\ &\le \liminf_{u,v\in H, \|u\|,\|v\|\to\infty} (\|Fu - v\|^2 + \|Kv + u\|^2)^{1/2} \\ &= \liminf_{\|z\|\to\infty} \|Tz\|. \end{aligned}$$

Thus by Theorem TZ we have that $N(T) \ne \emptyset$ and hence by Theorem 2.3 we obtain that $z_n = [u_n, v_n] \to z^* = [u^*, v^*] \in N(T)$, i.e., an element of Ω.

Theorem 2.5 Let H be a real Hilbert space. Let $F : D(F) \subseteq H \to H$, $K : D(K) \subseteq H \to H$ be bounded uniformly monotone mappings with $\mathcal{R}(F) \subseteq D(K)$, where $D(F)$ and $D(K)$ are closed convex subsets of H. Suppose that $N(T) := \{z^* \in D(T) : Tz^* = 0\} \ne \emptyset$. Let $\{\alpha_n\}$ and $\{\lambda_n\}$ be real sequences in (0, 1] satisfying the following conditions:

(i) $\lim_{n\to\infty} \lambda_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n\lambda_n = \infty$;
(iii) $\sum_{n=0}^{\infty} \alpha_n\lambda_n^2 < \infty$.

Let sequences $\{u_n\} \subseteq D(F)$ and $\{v_n\} \subseteq D(K)$ be generated from $u_0 \in D(F)$ and $v_0 \in D(K)$, respectively, by

$$\begin{cases} u_{n+1} = (1-\alpha_n)u_n + \alpha_n P_{D(F)}(u_n - \lambda_n(Fu_n - v_n)), \\ v_{n+1} = (1-\alpha_n)v_n + \alpha_n P_{D(K)}(v_n - \lambda_n(Kv_n + u_n)). \end{cases} \tag{2.17}$$

Then there exists $d > 0$ such that whenever $\lambda_n \le d$ for all $n \ge 0$, the sequence $\{z_n\}$ converges strongly to an element $\hat{z} = [\hat{u}, \hat{v}]$ of Ω with $N(T) = \{\hat{z}\}$ where $z_n = [u_n, v_n]$ for all $n \ge 0$.

Proof Since F and K are bounded uniformly monotone mappings, it is known that T is a bounded uniformly monotone mapping, that is, for each $z_1, z_2 \in D(T)$ there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle Tz_1 - Tz_2, z_1 - z_2 \rangle \ge \phi(\|z_1 - z_2\|).$$

Thus it is obvious that T is pseudomonotone. Also since $N(T) \ne \emptyset$, we get that $\Omega \ne \emptyset$. Next we divide the remainder of the proof into several steps.

Step 1. We claim that the sequences $\{u_n\}$ and $\{v_n\}$ conforming to (2.1) are well defined. Indeed, for initial point $z_0 := [u_0, v_0]$, define the sequence $\{z_n\}$ by

$$z_{n+1} := (1-\alpha_n)z_n + \alpha_n P_{D(F)\times D(K)}(z_n - \lambda_n T z_n). \tag{2.18}$$

As in Step 1 of the proof of Theorem 2.1, we can prove that the sequences $\{u_n\}$ and $\{v_n\}$ conforming to (2.17) are well defined.

Step 2. As in Step 2 of the proof of Theorem 2.1, we can prove that the sequence $\{z_n\}$ is bounded.

Step 3. We claim that the sequence $\{z_n\}$ converges strongly to an element $\hat{z} = [\hat{u}, \hat{v}]$ of Ω with $N(T) = \{\hat{z}\}$. Indeed, let $\hat{z} = [\hat{u}, \hat{v}] \in N(T)$ be arbitrary but fixed. utilizing the same method as in the reasoning of (2.4) and noticing that $T\hat{z} = 0$, we deduce that

$$\begin{aligned} \|z_{n+1} - \hat{z}\|^2 &\le \|z_n - \hat{z}\|^2 - 2\alpha_n\lambda_n\langle Tz_n, z_n - \hat{z}\rangle + \alpha_n\lambda_n^2\|Tz_n\|^2 \\ &\le \|z_n - \hat{z}\|^2 - 2\alpha_n\lambda_n\phi(\|z_n - \hat{z}\|) + \alpha_n\lambda_n^2\|Tz_n\|^2 \\ &\le (1 - 2\alpha_n\lambda_n\sigma(z_n, \hat{z}))\|z_n - \hat{z}\|^2 + \alpha_n\lambda_n^2 M_* \end{aligned} \tag{2.19}$$

for some constant $M_* > 0$ (since $\{z_n\}$ and $\{Tz_n\}$ are bounded), where

$$\sigma(z_n, \hat{z}) = \frac{\phi(\|z_n - \hat{z}\|)}{1 + \phi(\|z_n - \hat{z}\|) + \|z_n - \hat{z}\|^2}.$$

Then it follows from (2.19) that

$$\|z_{n+1} - \hat{z}\|^2 \le \|z_n - \hat{z}\|^2 + \alpha_n\lambda_n^2 M_*. \tag{2.20}$$

Now, put $\delta_n = 0$ and $b_n = \alpha_n\lambda_n^2 M_*$ for all $n \ge 0$. Then (2.20) can be rewritten as

$$\|z_{n+1} - \hat{z}\|^2 \le (1 + \delta_n)\|z_n - \hat{z}\|^2 + b_n.$$

Since $\sum_{n=0}^{\infty} \delta_n < \infty$ and $\sum_{n=0}^{\infty} b_n < \infty$, by Lemma 1.3 we know that $\lim_{n\to\infty} \|z_n - \hat{z}\|$ exists. Suppose $\lim_{n\to\infty} \|z_n - \hat{z}\| = \delta \ge 0$. We can prove that $\delta = 0$. Suppose on the contrary, $\delta > 0$. Let $N_0 \ge 0$ be an integer such that $\|z_n - \hat{z}\| \ge \frac{\delta}{2}, \forall n \ge N_0$. Then

$$\phi(\|z_n - \hat{z}\|) \ge \phi\left(\frac{\delta}{2}\right) > 0, \quad \forall n \ge N_0.$$

Since $\|z_n - \hat{z}\| \leq D, \forall n \geq 0$ for some $D \geq 0$, from (2.19) we obtain that

$$\begin{aligned} 2\alpha_n\lambda_n \frac{\phi(\frac{\delta}{2})(\frac{\delta}{2})^2}{1+\phi(D)+D^2} &\leq 2\alpha_n\lambda_n\sigma(z_n, \hat{z})\|z_n - \hat{z}\|^2 \\ &\leq \|z_n - \hat{z}\|^2 - \|z_{n+1} - \hat{z}\|^2 + \alpha_n\lambda_n^2 M_* \end{aligned} \tag{2.21}$$

for all $n \geq N_0$. Since $\sum_{n=0}^{\infty} \alpha_n\lambda_n^2 < \infty$, from (2.21) we derive $\sum_{n=0}^{\infty} \alpha_n\lambda_n < \infty$, which contradicts the condition $\sum_{n=0}^{\infty} \alpha_n\lambda_n = \infty$. Thus, $\delta = 0$. This implies that $\{z_n\}$ converges strongly to $\hat{z}$. According to the uniqueness of the limit, $N(T)$ is a singleton.

Acknowledgements The authors would like to express their sincere appreciation to the referees for their valuable comments and suggestions which improved the original manuscript much. This research was partially supported by the Teaching and Research Award Fund for Outstanding Young Teachers in Higher Education Institutions of MOE, China and the Dawn Program Foundation in Shanghai. This research was partially supported by a grant from the National Science Council.

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